

Inhomogeneous distribution of particles and temperature in a self-gravitating system

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Self-gravitating systems are non-equilibrium *a priori*. A new approach is proposed, which employs a non-equilibrium statistical operator taking into account inhomogeneous distributions of particles and temperature. The method involves the saddle-point procedure to find the dominant contributions to the partition function and thus to obtain all thermodynamic parameters of the system. Probable peculiar features in the behavior of the self-gravitating system are considered for various conditions. The equation of state for self-gravitating system has been determined. A new length of the statistical instability and parameters of the spatially inhomogeneous distribution of particles and temperature are obtained for a real gravitational system.

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INTRODUCTION

The study of self-gravitating systems is of fundamental physical importance. Such system are suitable for testing the ideas on the statistical description of systems with by long-range interaction. Moreover, a self-gravitating system concerns a general problem that has been studied for a long time [1] and turned out to be more complicated than other many-body system. Thus, the self-gravitating system provides a model of fundamental interest for which ideas of statistical mechanics and thermodynamics can be tested and developed [2]. The statistical description of the system can be helpful for the problems of astrophysics [3],[4] and for the development of general approaches to the investigation of structure formation in various physical situations.

The statistical mechanics of the systems under consideration turns out to be very different from the study of other, more familiar, many-body systems. The fundamental difference is that the concept of equilibrium is not always well defined and these systems manifest nontrivial behavior with under phase transitions associated with the gravitational collapse. The standard methods of the equilibrium statistical mechanic cannot be applied to the study of self-gravitating systems since in this case the thermodynamic ensembles are not equivalent, i.e., the negative specific heat [5] occurring in the micro-canonical ensemble does not exist in the canonical description [6]. In the micro-canonical ensemble, the self-collapse corresponds to the "gravity-thermal catastrophe" while in the canonical ensemble it is associated with the "isothermal collapse" [7]. Energy of an isolated self-gravitating system is conserved and fundamentally, only the concept of micro-canonical ensemble makes natural sense. In particular, inasmuch as energy is not-additive, the canonical ensemble cannot be applied to the study of systems with long-range interactions.

Two approaches (statistical and thermodynamical) have been developed to find the equilibrium states of the self-gravitating system and to describe probable phase transitions [6], [7]. The collapse in such systems starts as a spatially inhomogeneous distribution of particles in all the system at once.

A very important question is whether the inhomogeneous distribution of particles can be regarded as an equilibrium state. Phase transitions in such systems require the description in terms of the mean-field thermodynamics [7]. It is generally believed that the mean field theory is exact for equilibrium systems. Since in the mean field theory any thermodynamic function depends only on dimensionless combinations of thermodynamic variables, it follows from the article [8] and the book [9] that the thermodynamic limit of the system does not exist. The non-existence of the thermodynamic limit implies that the thermodynamic potentials do not scale properly with the number of particles and thus the thermodynamic functions, i.e., temperature, diverge. Nevertheless, we can take the ordinary thermodynamic limit and then employ the usual thermodynamic tools by first regularizing the long-distance behavior of the gravitational potential and then introducing a very large screening length. The system is then thermodynamically stable and the thermodynamic limit does exist [10].

Formation of spatially inhomogeneous distributions of interacting particles is a typical problem in condensed matter physics and requires non-conventional statistical description in the case of particles involved in the gravitational interaction with regard for an arbitrary spatially inhomogeneous particle distribution. The statistical description should employ the procedure to find the dominant contributions to the partition function and to avoid entropy divergences for infinite system volume.

A non-conventional method was proposal in [11], [12], [13]. It employs the Hubbard-Stratonovich representation of the statistical sum [14] that is extended to a system with gravitational interaction in order to find a solution for the particle distribution making no use of spatial box restrictions. It is important that this solution has no divergences in

the thermodynamic limits. For this purpose the saddle point approximation is applied, which takes into account the conservation of the number of particles in the limiting space and yields a nonlinear equation. The three-dimensional solution of this nonlinear equation selects the states whose contributions in the partition function are dominant. The partition function for the case of homogeneous particle distribution, as well as for the case of inhomogeneous distribution, was obtained in [12], [15]. This approach, however, describes only the condition for the formation of probable inhomogeneous distributions of gravitating particles.

Systems with long-range interactions, such as self-gravitating system, do not relax to the usual Boltzmann-Gibbs thermodynamic equilibrium, but get trapped in the quasi-stationary states whose lifetimes diverge as the number of particles increases. A theory was proposed that provides a quantitative prediction of the instability threshold for the spontaneous symmetry breaking for a class of d -dimensional systems [16]. The non-equilibrium stationary states of such systems were described in the paper [17] where the three-dimensional systems were shown to be trapped in non-equilibrium quasi-stationary states rather than evolve to the thermodynamic equilibrium [17], mainly because the self-gravitating systems exist in a highly nonequilibrium states and the time of relaxation to the equilibrium state is very long. The homogeneous particle distribution in a self-gravitating system is not stable. Particle distributions in such systems are spatially inhomogeneous, from the very beginning. Therefore, the system brakes on a complex of inhomogeneous clusters, which collapse to more condensed states. The description of typical behavior of a self-gravitating system should be specific for various equilibrium ensembles.

Some attempts to include particle distribution inhomogeneity into consideration have been made [23]-[25], however, the solution has not been found up to now. The reason is that, with the inhomogeneity included, the chemical potential depends on the space variables and the relevant equations of state should interconnect temperature and density. Hence the temperature, as a thermodynamic parameter, has to depend on the space coordinates too. The concentration-dependence of temperature and is found in order to obtain stable solutions for the gravitational formation of stars [7]. This approach seems to be inconsistent, since the state equation should follow from the definition of the partition function which, however, is unknown for spatially inhomogeneous systems [18], [9]. Therefore, there is a dilemma, whether to employ the postulates of equilibrium statistical mechanics and obtain only instability criteria or not to try to take into account the spatial inhomogeneity and use a different approach. Such inhomogeneous distributions of particles, temperature, and chemical potential can be accounted for in the non-equilibrium statistical operator approach [19] with allowance for probable local changes of the thermodynamic parameters. This system is non-equilibrium and thus inhomogeneous particle distribution can justify the inhomogeneous distribution of temperature, chemical potential, and other thermodynamic parameters.

In this article we propose a new approach in terms of the non-equilibrium statistical operator [19] that is more suitable for the description of gravitational systems. The equation of state and all thermodynamic characteristics needed are defined by equations which govern the greatest contributions to the partition function. Thus, there is no need to introduce additional hypothesis about the density-dependence of temperature. This dependence is obtained by solving corresponding thermodynamic relations which describe the extremums of the non-equilibrium partition function. The possible spatially inhomogeneous distributions of particles and temperature are obtained for simple cases. For the equilibrium case, the well-known result [20],[21] for the partition function is reproduced. This approach is shown to describe the inhomogeneous particle distribution and to determine the thermodynamic parameters in the self-gravitating system. The main idea of this paper is to provide a detailed description of self-gravitating systems in terms of the principles of non-equilibrium statistical mechanics and to obtain distributions of particles and temperature for fixed number of particles and energy of the system.

NON-EQUILIBRIUM STATISTICAL SUM

Phenomenological thermodynamics is based on the conservation laws for the average values of physical parameters, i.e., the number of particles, energy, and momentum. Statistical thermodynamics of non-equilibrium systems is also based on the conservation laws, however, for the dynamic variables rather than their average values. It presents local conservation laws for the dynamic variables. In order to find thermodynamic functions of a non-equilibrium system, we have to use the presentation of relevant statistical ensembles taking into account the non-equilibrium states of these systems. The conception of Gibbs ensembles can provide a description of non-equilibrium stationary states of the system. In this case we can define a non-equilibrium ensemble as a totality of systems that can be contained in the same stationary external condition. This system possesses the character of contact similar to a thermostat and possesses all the probable values of macroscopic parameters compatible with the given conditions. Local equilibrium stationary distributions are formed in systems under similar stationary external conditions. If the external condition depends on time, the relevant local equilibrium distribution is not stationary. In order to determine

a local equilibrium ensemble exactly, we have to determine the distribution function or the statistical operator of the system [19]. Finally, we remind the reader that stable states of classical self-gravitating particles are only metastable because they correspond to the local maxima of the thermodynamic potential. This thermodynamic potential is the local entropy whose extremum determines the behavior of the system.

Under the assumption that non-equilibrium states of the system can be written in terms of the inhomogeneous distribution energy, $H(\mathbf{r})$, and the number of particles (density), $n(\mathbf{r})$, the local equilibrium distribution function for a classical system can write in the form [19]

$$f_l = Q_l^{-1} \exp \left\{ - \int (\beta(\mathbf{r})H(\mathbf{r}) - \eta(\mathbf{r})n(\mathbf{r})) d\mathbf{r} \right\} \quad (1)$$

where

$$Q_l = \int D\Gamma \exp \left\{ - \int (\beta(\mathbf{r})H(\mathbf{r}) - \eta(\mathbf{r})n(\mathbf{r})) d\mathbf{r} \right\} \quad (2)$$

determine the statistical operator of the local equilibrium distribution. The integration in this formula should be performed over the whole phase space of the system. It should be noted that in the case of local equilibrium distribution, the Lagrange multipliers $\beta(\mathbf{r})$ and $\eta(\mathbf{r})$ are functions of a spatial point. The density of particles can be presented in the standard form, i.e.,

$$n(\mathbf{r}) = \sum_i \delta(\mathbf{r} - \mathbf{r}_i) \quad (3)$$

The local equilibrium distribution can be introduced provided the relaxation time in the whole system is greater than the relaxation time in a local macroscopic region contained in this system.

Having determined the non-equilibrium statistical operator we can obtain all the thermodynamic parameters of the non-equilibrium system. To do this we have to derive a thermodynamic relation for inhomogeneous systems. The variation of the statistical operator by the Lagrange multipliers yields the required thermodynamic relation in the form [19]:

$$- \frac{\delta \ln Q_l}{\delta \beta(\mathbf{r})} = \langle H(\mathbf{r}) \rangle_l \quad (4)$$

and

$$\frac{\delta \ln Q_l}{\delta \eta(\mathbf{r})} = \langle n(\mathbf{r}) \rangle_l \quad (5)$$

This relation is a natural general extension of the well-known relation for equilibrium systems to the case of inhomogeneous system. Conservation of the number of particles and energy in the system is given by

$$\int n(\mathbf{r}) d\mathbf{r} = N \quad (6)$$

and

$$\int H(\mathbf{r}) d\mathbf{r} = E \quad (7)$$

To continue the statistical description of non-equilibrium systems, we have to find the Hamiltonian of the system. In the general case, the Hamiltonian of a system of interacting particles is given by

$$H = \sum_i \frac{p_i^2}{2m} + \frac{1}{2} \sum_{i,j} W(\mathbf{r}_i \mathbf{r}_j) \quad (8)$$

The energy of gravitational interaction energy can be written in the well-known form

$$W(\mathbf{r}, \mathbf{r}') = \frac{Gm^2}{|\mathbf{r} - \mathbf{r}'|} \quad (9)$$

where G is the gravitational constant and m is the mass of a separate particle. In what follows we use only the density of energy which for a self-gravitating system can be written in the form

$$H(\mathbf{r}) = \frac{p^2(\mathbf{r})}{2m}n(\mathbf{r}) + \frac{1}{2} \int W(\mathbf{r}, \mathbf{r}')n(\mathbf{r})n(\mathbf{r}')d\mathbf{r}' \quad (10)$$

This energy density of a system can be used if we divide the space into fragments with equal masses and consider motion in the phase space of an incompressible gravitational fluid. This model is valid for collision-less systems and particles with gravitational interaction provide a very good example of such system.

The non-equilibrium statistical operator of a self-gravitating system can be written as

$$Q_l = \int D\Gamma \exp \left\{ - \int \left(\beta(\mathbf{r}) \frac{p^2(\mathbf{r})}{2m} - \eta(\mathbf{r}) \right) n(\mathbf{r}) d\mathbf{r} - \frac{1}{2} \int W(\mathbf{r}, \mathbf{r}')n(\mathbf{r})n(\mathbf{r}')d\mathbf{r}d\mathbf{r}' \right\} \quad (11)$$

Integration over the phase space is given by $D\Gamma = \frac{1}{(2\pi\hbar)^3} \prod_i dr_i dp_i$.

In order to perform formal integration in the second part of this paper, we introduce additional field variables in terms of the theory of Gaussian integrals [14], [13], i.e.,

$$\exp \left\{ - \frac{1}{2} \int \beta(\mathbf{r}) W(\mathbf{r}, \mathbf{r}') n(\mathbf{r}) n(\mathbf{r}') d\mathbf{r} d\mathbf{r}' \right\} = \int D\varphi \exp \left\{ - \frac{1}{2} \int W^{-1}(\mathbf{r}, \mathbf{r}') \varphi(\mathbf{r}) \varphi(\mathbf{r}') d\mathbf{r} d\mathbf{r}' - \int \sqrt{\beta(\mathbf{r})} \varphi(\mathbf{r}) n(\mathbf{r}) d\mathbf{r} \right\} \quad (12)$$

where $D\varphi = \frac{\prod_s d\varphi_s}{\sqrt{\det 2\pi\beta W(\mathbf{r}, \mathbf{r}')}} d\varphi$ and $W^{-1}(\mathbf{r}, \mathbf{r}')$ is the inverse operator that satisfies the condition $W^{-1}(\mathbf{r}, \mathbf{r}')W(\mathbf{r}', \mathbf{r}'') = \delta(\mathbf{r} - \mathbf{r}'')$. Now the field variable $\varphi(\mathbf{r})$ contains the same information as the original distribution function, i.e., complete information about probable spatial states of the system. The inverse operator $W^{-1}(\mathbf{r}, \mathbf{r}')$ of the gravitational interaction in the continuum limit should be treated in the operator sense, i.e.,

$$W^{-1}(\mathbf{r}, \mathbf{r}') = - \frac{1}{4\pi G m^2} \Delta_{\mathbf{r}} \delta(\mathbf{r} - \mathbf{r}') \quad (13)$$

where $\Delta_{\mathbf{r}}$ is the Laplace operator in the real space. The statistical operator reduces now to the form

$$Q_l = \int D\Gamma \int D\varphi \exp \left\{ - \int \left(\beta(\mathbf{r}) \frac{p^2(\mathbf{r})}{2m} - \eta(\mathbf{r}) - \sqrt{\beta(\mathbf{r})} \varphi(\mathbf{r}) \right) n(\mathbf{r}) d\mathbf{r} - \frac{1}{8\pi m^2 G} \int (\nabla \varphi(\mathbf{r}))^2 d\mathbf{r} \right\} \quad (14)$$

This functional integral can be integrated over the phase space. Making use of the definition of density we can rewrite the non-equilibrium statistical operator as

$$Q_l = \int D\varphi \int \frac{1}{(2\pi\hbar)^3 N!} \prod_i dr_i dp_i \xi(\mathbf{r}_i) \exp \left\{ - \left(\beta(\mathbf{r}_i) \frac{p_i^2}{2m} - \sqrt{\beta(\mathbf{r}_i)} \varphi(\mathbf{r}_i) \right) - \frac{1}{8\pi m^2 G} \int (\nabla \varphi(\mathbf{r}))^2 d\mathbf{r} \right\} \quad (15)$$

where $\xi(\mathbf{r}) \equiv \exp \eta(\mathbf{r})$ is a new variable that can be interpreted as chemical activity. Now we can perform integration over the momentum. The non-equilibrium statistical operator is given by

$$Q_l = \int D\varphi \exp \left\{ - \frac{1}{8\pi m^2 G} \int (\nabla \varphi(\mathbf{r}))^2 d\mathbf{r} \right\} \frac{1}{N!} \prod_i \int dr_i \xi(\mathbf{r}_i) \left(\frac{2\pi m}{\hbar^3 \beta(\mathbf{r}_i)} \right)^{\frac{3}{2}} \exp \left(\sqrt{\beta(\mathbf{r}_i)} \varphi(\mathbf{r}_i) \right) \quad (16)$$

$$Q_l = \int D\varphi \exp \left\{ - \frac{1}{8\pi m^2 G} \int (\nabla \varphi(\mathbf{r}))^2 d\mathbf{r} \right\} \sum_N \frac{1}{N!} \int d\mathbf{r} \xi(\mathbf{r}) \left(\frac{2\pi m}{\hbar^3 \beta(\mathbf{r})} \right)^{\frac{3}{2}} \exp \left(\sqrt{\beta(\mathbf{r})} \varphi(\mathbf{r}) \right)^N \quad (17)$$

Now it reduces to the simple form

$$Q_l = \int D\varphi \exp \left\{ \int \left[- \frac{1}{8\pi m^2 G} (\nabla \varphi(\mathbf{r}))^2 + \xi(\mathbf{r}) \left(\frac{2\pi m}{\hbar^3 \beta(\mathbf{r})} \right)^{\frac{3}{2}} \exp \sqrt{\beta(\mathbf{r})} \varphi(\mathbf{r}) \right] d\mathbf{r} \right\} \quad (18)$$

For constant temperature β and absolute chemical activity ξ , the statistical operator fully reproduces the equilibrium canonical partition function [12], [22]. In our general case, the non-equilibrium statistical operator can be rewritten in the form

$$Q_l = \int D\varphi \exp \{ -S(\varphi(\mathbf{r}), \xi(\mathbf{r}), \beta(r)) \} \quad (19)$$

where the effective non-equilibrium "local entropy" is given by

$$S(\varphi(\mathbf{r}), \xi(\mathbf{r}), \beta(r)) = \int \left[\frac{1}{8\pi m^2 G} (\nabla \varphi(\mathbf{r}))^2 - \xi(\mathbf{r}) \left(\frac{2\pi m}{\hbar^2 \beta(\mathbf{r})} \right)^{\frac{3}{2}} \exp \sqrt{\beta(r)} \varphi(\mathbf{r}) \right] d\mathbf{r} \quad (20)$$

The statistical operator makes it possible to use the efficient methods developed in the quantum field theory without any additional restrictions for the integration over the field variables or the perturbation theory. The functional $S(\varphi(\mathbf{r}), \xi(\mathbf{r}), \beta(r))$ depends on the distribution of the field variables $\varphi(\mathbf{r})$, the chemical activity $\xi(\mathbf{r})$, and the inverse temperature $\beta(\mathbf{r})$. Now the saddle-point method can be employed to find the asymptotic value of the statistical operator Q_l for N to ∞ ; the dominant contribution is given by the states which satisfy the extremum conditions for the functional. It is not difficult to see that the saddle-point equation represents the thermodynamic relation and may be reduced to an equation for the field variable, i.e.,

$$\frac{\delta S}{\delta \varphi(\mathbf{r})} = 0 \quad (21)$$

with the normalization condition being given by

$$\frac{\delta S}{\delta(\eta(\mathbf{r}))} = - \int \frac{\delta S}{\delta(\xi(\mathbf{r}))} \xi(\mathbf{r}) d\mathbf{r} = N \quad (22)$$

, and for the energy conservation in the system, i.e.,

$$- \int \frac{\delta S}{\delta(\beta(\mathbf{r}))} \xi(\mathbf{r}) d\mathbf{r} = E \quad (23)$$

The solution of this equation completely determines all the thermodynamic parameters and describes the general behavior of a self-gravitating system both for spatially homogeneous and inhomogeneous particle distributions. The above set of equations in principle solves the many-particle problem in the thermodynamic limit. The spatially inhomogeneous solution of these equations corresponds to the distribution of interacting particles. Such inhomogeneous behavior is associated with the nature and intensity of the interaction. In other words, accumulation of particles in a finite spatial region (cluster formation) reflects the spatial distribution of the field, activity, and temperature. It is very important to note that this approach is the only one that makes it possible to take into account the inhomogeneity of temperature distribution that may depend on the spatial distribution of particle in the system. In other approaches, the dependence of temperature on a spatial point is introduced through the polytropic dependence of temperature on particle density in the equation of state [7]. In the present approach, this dependence follows from the necessary thermodynamic condition and can be found for various particle distributions.

Now we derive the saddle-point equation for the extremum of the local entropy functional $S(\varphi, \xi, \beta)$. The equation for the field variable $\frac{\delta S}{\delta \varphi} = 0$ yields

$$\frac{1}{r_m} \Delta \varphi(\mathbf{r}) + \xi(\mathbf{r}) \left(\frac{2\pi m}{\hbar^2 \beta(\mathbf{r})} \right)^{\frac{3}{2}} \sqrt{\beta(r)} \exp(\sqrt{\beta(r)} \varphi(\mathbf{r})) = 0 \quad (24)$$

where the notation $r_m \equiv 4\pi G m^2$ is introduced. The normalization condition may be written as

$$\int \xi(\mathbf{r}) \left(\frac{2m}{\hbar^2 \beta(\mathbf{r})} \right)^{\frac{3}{2}} \exp(\sqrt{\beta(r)} \varphi(\mathbf{r})) d\mathbf{r} = N \quad (25)$$

and the equation for the energy conservation in the system is given by

$$\frac{3}{2} \int \left(\frac{2\pi m}{\hbar^2 \beta(\mathbf{r})} \right)^{\frac{3}{2}} \frac{\xi(\mathbf{r})}{\beta(\mathbf{r})} (3 - \sqrt{\beta(r)} \varphi(\mathbf{r})) \exp(\sqrt{\beta(r)} \varphi(\mathbf{r})) d\mathbf{r} = E \quad (26)$$

To draw more information on the behavior of a self-gravitating system, we introduce new variables. The normalization condition $\int \rho(\mathbf{r}) d\mathbf{r} = N$ yields the definition for the density function, i.e.,

$$\rho(\mathbf{r}) \equiv \left(\frac{2\pi m}{\hbar^2 \beta(\mathbf{r})} \right)^{\frac{3}{2}} \xi(\mathbf{r}) \exp(\sqrt{\beta(r)} \varphi(\mathbf{r})) \quad (27)$$

which reduces the equation to a simpler form. The equation for the field variable is given by

$$\Delta\varphi(\mathbf{r}) + r_m\sqrt{\beta(\mathbf{r})}\rho(\mathbf{r}) = 0 \quad (28)$$

The equation for energy conservation takes the form

$$\frac{1}{2} \int \frac{\rho(\mathbf{r})}{\beta(\mathbf{r})} (3 - \sqrt{\beta(\mathbf{r})}\varphi(\mathbf{r})) d\mathbf{r} = E \quad (29)$$

The equation thus obtained cannot be solved in the general case, but it is possible to analyze many cases of the behavior of a self-gravitating system under various external conditions. In what follows we write the chemical activity in terms of the chemical potential $\xi(\mathbf{r}) = \exp(\mu(\mathbf{r})\beta(\mathbf{r}))$. Having differentiated the equation for energy conservation over the volume, we obtain an interesting relation for the chemical potential, i.e.,

$$\frac{1}{2} \frac{\rho(\mathbf{r})}{\beta(\mathbf{r})} (3 - \sqrt{\beta(\mathbf{r})}\varphi(\mathbf{r})) = \frac{\delta E}{\delta V} \frac{\delta V}{\delta N} = \mu(\mathbf{r})\rho(\mathbf{r}) \quad (30)$$

which yields the chemical potential to be given by

$$\mu(\mathbf{r})\beta(\mathbf{r}) = \frac{3}{2} - \frac{1}{2}\sqrt{\beta(\mathbf{r})}\varphi(\mathbf{r}) \quad (31)$$

Within the context of the expression for the density and the definition of thermal de-Broglie wavelength and the gravitation length, i.e.,

$$\Lambda^{-1}(\mathbf{r}) = \left(\frac{2m}{\hbar^2\beta(\mathbf{r})} \right), R_g(\mathbf{r}) = 2\pi Gm^2\beta(\mathbf{r}) \quad (32)$$

we can rewrite all the equations and the normalization condition in terms of density and temperature. Thus we have

$$\Delta \left(\frac{\Lambda^3(\mathbf{r})\rho(\mathbf{r})}{\sqrt{\beta(\mathbf{r})}} \right) + \frac{R_g(\mathbf{r})}{\sqrt{\beta(\mathbf{r})}}\rho(\mathbf{r}) = 0 \quad (33)$$

and the chemical potential reduces to

$$\mu(\mathbf{r})\beta(\mathbf{r}) = \frac{3}{2} - \ln(\Lambda^3(\mathbf{r})\rho(\mathbf{r})) \quad (34)$$

In this approach we can obtain too the equation of state for self-gravitating system if use the thermodynamic relation $P = -\frac{1}{\beta} \frac{\delta S}{\delta V}$ in the case energy conservation E . In our case, in the definition of “local entropy” can use the relation $(\nabla\varphi(\mathbf{r}))^2 = \nabla(\varphi(\mathbf{r})\nabla\varphi(\mathbf{r})) - \varphi(\mathbf{r})\Delta\varphi(\mathbf{r})$ after that provide the integration on all volume. First part of integration can present as surface integral where $\varphi(\mathbf{r}) = 0$ on the integration surface. After that we can present the “local entropy as

$$S = \int \left[-\frac{1}{8\pi m^2 G} \varphi(\mathbf{r})\Delta\varphi(\mathbf{r}) - \xi(\mathbf{r}) \left(\frac{2\pi m}{\hbar^2\beta(\mathbf{r})} \right)^{\frac{3}{2}} \exp \sqrt{\beta(\mathbf{r})}\varphi(\mathbf{r}) \right] d\mathbf{r} \quad (35)$$

which can rewrite, using the definition of density of particle, in the the form

$$S = \int [-\rho(\mathbf{r}) \ln(\Lambda^3(\mathbf{r})\rho(\mathbf{r})) - \rho(\mathbf{r})] d\mathbf{r} \quad (36)$$

The local equation of state can present as

$$P(\mathbf{r})\beta(\mathbf{r}) = \rho(\mathbf{r})(1 - \ln(\Lambda^3(\mathbf{r})\rho(\mathbf{r}))) = \rho(\mathbf{r}) \left(\mu(\mathbf{r})\beta(\mathbf{r}) - \frac{1}{2} \right) \quad (37)$$

In the classical case $\Lambda^3(\mathbf{r})\rho(\mathbf{r}) \ll 1$ and $P\beta \equiv \rho$ but have the multiple which logarithmic dependence from density of particle. Only in the case $\Lambda^3(\mathbf{r})\rho(\mathbf{r}) = 1$ we obtain the equation of state for ideal gas. In the case of the ideal gas we obtain usual equation of state, because in this case $\varphi(\mathbf{r}) = 0$ and $P\beta = \rho$ as result absent of interaction. In the case of ideal gas $\mu(\mathbf{r})\beta(\mathbf{r}) = \frac{3}{2}$ and the equation of state reproduce the equation of state of the ideal gas. In this case the energy of system equal $E = \frac{3}{2}NkT$ that satisfy the previous obtained results. In the next section we find the classical distributions of particles for various inner and external conditions.

PARTICLE AND TEMPERATURE DISTRIBUTIONS IN A SELF-GRAVITATING SYSTEM

Homogeneous distribution of particles

a) First of all we consider the equilibrium case, with all the parameters being independent of space coordinates. In this case, the energy and total number of particles are fixed and, moreover, the temperature and the chemical potential do not change in space. Thus the equation for the particle concentration

$$\Delta \left(\frac{\Lambda^3(\mathbf{r})\rho(\mathbf{r})}{\sqrt{\beta(\mathbf{r})}} \right) + \frac{R_g(\mathbf{r})}{\sqrt{\beta(\mathbf{r})}}\rho(\mathbf{r}) = 0 \quad (38)$$

leads to a simple condition $\sqrt{\beta}\rho(\mathbf{r}) = 0$ that can be realized only for $T \rightarrow \infty$. The particle distribution in a self-gravitating system can be homogeneous only for very high temperatures.

b) Another interesting case is when only particle density depends on the coordinate while the temperature is fixed. In this case the equation for the density takes the form

$$\Delta (\ln \Lambda^3 \rho(\mathbf{r})) + R_g \rho(\mathbf{r}) = 0 \quad (39)$$

and can be transformed to

$$\Delta (\ln \rho(\mathbf{r})) + R_g \rho(\mathbf{r}) = 0. \quad (40)$$

The latter equation has an exact solution $\rho(\mathbf{r}) = \frac{2}{R_g r^2}$ but the normalization condition holds only for the case of a fixed box with size $R = \frac{NGm^2}{4kT}$, fixed energy $E = NkT$, and the changes of the chemical-potential density within the box given by $\mu = kT(\frac{3}{2} - \frac{2\Lambda^3}{4kTR_g r^2})$.

As follows from the equation for constant temperature, the homogeneous distribution of particles is unstable. The homogeneous distribution of particle $\rho(\mathbf{r}) = \rho + \delta\rho(\mathbf{r})$ yields an equation for density fluctuations, i.e.,

$$\Delta \delta\rho(\mathbf{r}) + R_g \rho \delta\rho(\mathbf{r}) = 0 \quad (41)$$

that reproduces the Helmholtz equation. The general solution of the wave equation is the unstable radial distribution $\delta\rho(\mathbf{r}) = \frac{e^{ipkr}}{r}$ with the wave number $k = \sqrt{2\pi Gm^2\beta\rho}$ that implies that the wavelength of the instability is half as long as the Jeans length. It is the statistical length of the instability of particle distribution in the system.

The concept of Jeans gravitational instability is discussed within the framework of non-extensive statistics and its associated kinetic theory [26]. A simple analytical formula generalizing the Jeans criterion is derived by assuming that the unperturbed collisionless gas is kinetically described by the class of power-law velocity distributions. It is found that the critical values of the wavelength and mass depend explicitly on the non-extensive parameter. The instability condition is weakened as the system becomes unstable even for wavelengths of the disturbance smaller than the standard Jeans length.

The recent discoveries of extrasolar giant planets, coupled with refined models of the compositions of Jupiter and Saturn, prompt a reexamination of the theories of giant planet formation. An alternative to the favored core accretion hypothesis is examined in [27], the conclusion is that the gravitational instability in the outer solar nebula leads to the formation of giant planets. Three-dimensional hydrodynamic calculations predict formation with locally isothermal or adiabatic thermodynamics. The gravitational instability appears to be capable of forming giant planets [28]. Our results can help to explain the data of astrophysical observations in the sense that the different length of the instability in a self-gravitation system is associated with the alternative description of the situation. Thus we can conclude that particle distributions cannot be homogeneous for constant temperatures in the system. Thus we have to find real distributions of particles and temperature in the system

Inhomogeneous distributions of particles and temperature in a self-gravitating system

In the general case, particle distributions in self-gravitating systems are inhomogeneous. Inhomogeneous distribution of particles gives rise to the long-range gravitational interaction. Now we consider the non-equilibrium description of a self-gravitating system and take into account probable spatially inhomogeneous distributions of particles and

temperature. We introduce a new variable $\psi = \Lambda^3(\mathbf{r})\rho(\mathbf{r})$, then the equation for the density is simplified, i.e., we have

$$\Delta \left(\frac{\ln \psi}{\sqrt{\beta(\mathbf{r})}} \right) + \frac{R_g(\mathbf{r})}{\sqrt{\beta(\mathbf{r})}\Lambda^3(\mathbf{r})} \psi = 0 \quad (42)$$

The solution of this equation provides a complet non-equilibrium statistical description of a self-gravitating system. General exact solutions of this non-linear equation are unknown. In what follows we propose a way to solve this equation.

a) First of all we can find a more general solution of the problem. In the three-dimensional case the action of the Laplace operator can be presented in the form

$$\Delta \left(\frac{\ln \psi}{\sqrt{\beta(\mathbf{r})}} \right) = \frac{1}{\sqrt{\beta}} \left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) \ln \psi - \frac{\ln \psi}{\sqrt{\beta^3}} \left(\frac{d^2 \beta}{dr^2} + \frac{2}{r} \frac{d\beta}{dr} - \frac{3}{2\beta} \left(\frac{d\beta}{dr} \right)^2 \right) - \frac{1}{\sqrt{\beta^3}} \frac{d \ln \psi}{dr} \frac{d\beta}{dr} \quad (43)$$

The solution for the temperature is given by $\beta = \gamma^3 r^n$ and it is not difficult to see that for $n = 2$ we obtain only an equation for ψ , i.e.,

$$\frac{d^2 \ln \psi}{dr^2} + \frac{a_m}{B\gamma r} \psi = 0 \quad (44)$$

that can be rewritten in terms of the new variable $\bar{r}^2 = r$, i.e.,

$$\frac{d}{d\bar{r}} \left(\frac{1}{\psi} \frac{d\psi}{d\bar{r}} \right) + \frac{4a_m}{B\gamma} \psi = 0 \quad (45)$$

We multiply this equation by $\frac{1}{\psi} \frac{d\psi}{d\bar{r}}$ and calculate the first integral of the equation thus obtained. It is given by

$$\left(\frac{1}{\psi} \frac{d\psi}{d\bar{r}} \right)^2 + \frac{4a_m}{B\gamma} \psi = \Delta \quad (46)$$

and the exact solution can be written as

$$\psi = \frac{\Delta}{\frac{8a_m}{B\gamma} \sinh^2 \sqrt{\frac{\Delta r}{4}}} \quad (47)$$

Using the latter definition, we find the exact solution for the inhomogeneous particle distribution to be given by

$$\rho(\mathbf{r}) = \frac{\Delta}{8a_m \gamma^2 r^3 \sinh^2 \sqrt{\frac{\Delta r}{4}}} \quad (48)$$

and thus obtain good assumptions concerning the behavior at long distances from the center of the inhomogeneous particle distribution. This behavior related with result early obtained in articles [23]-[25] where was use Boltzmann equation for distribution function for spherical isolate stellar system. The distribution of particles is inhomogeneous for the size $R = \frac{1}{4\Delta}$ and divergent to center $\rho(\mathbf{r}) = \frac{1}{2a_m \gamma^2 r^4}$. In this case the energy of system are conserved. However, we do not know the coefficients. Thus we propose the approach given below. If particles are concentrated at short distances and their concentration is very high, the crucial factor is the quantum effect for which our approach is inapplicable. The relation between critical temperature and particle concentration in this quantum case is determined by a natural condition

$$\Lambda^3(\mathbf{r})\rho(\mathbf{r}) = \left(\frac{\hbar^2 \beta_c}{2me} \right)^{\frac{3}{2}} \rho_c = 1 \quad (49)$$

This relation along with the formula for the conservation of the number of particles, $\frac{4\pi}{2a_m R_c} = N$, determine all the required parameters, i.e., the critical distance $R_c = \frac{\hbar^2}{ma_m N^{\frac{1}{3}}}$, the coefficient $\gamma^2 = \frac{2\pi me}{\hbar^2 N^{\frac{2}{3}}}$, the critical temperature $\beta_c = \gamma^2 R_c^2$, and the concentration $\rho_c = \frac{1}{2a_m R_c^3}$. The energy of the system is in this case given by $E = \frac{3}{2} N k T$, i.e., is equal to the energy of a free particle! In this part we present the general solution for the classical particle distribution

at long distances from the center of an inhomogeneous cluster of condensed matter that is subject to the laws of quantum physics. In all the cases, our solution holds under the condition of classical physics, $\Lambda^3(\mathbf{r})\rho(\mathbf{r}) \ll 1$.

b) In this part we describe the system with $\Lambda^3(\mathbf{r})\rho(\mathbf{r}) = \alpha = \text{const} \ll 1$. In this case we can determine only the behavior of temperature that is governed by the equation

$$\Delta \left(\frac{1}{\sqrt{\beta(\mathbf{r})}} \right) + \frac{a_m e^\alpha}{B \ln \alpha} \frac{1}{\beta} = 0 \quad (50)$$

Similarly to the previous case, we write the solution of this equation in the form $\beta = \gamma^{-2} r^{-2n}$ and thus find that it holds for $n = -2$, i.e., the temperature is changed as $kT = \gamma^2 r^{-4}$, the concentration is changed as $\rho = A r^{-6}$ and normalization conditions for the conservation of particle number and energy are satisfied. The limiting behavior of the concentration and temperature, similarly to the previous part, provides a suitable solution of the problem behavior for this special case. Finally, we make an attempt to present an arbitrary solution in the general case of space-coordinate-dependences of the concentration and temperature. This equation describes any problem associated with inhomogeneous distributions of particles, temperature, and concentration in a self-gravitating system. Indeed, though the equation cannot be solved in the general case, it provides a possibility to analyze many cases of the behavior of a self-gravitating system under various external condition. We can consider many realistic distributions of concentration, temperature, and field in a gravitational system but should not use the equation of state, which must exist as a condition related with this equation. In a real system, the temperature cannot be related to the particle distribution. It is a thermodynamic parameter that determines a condition for the behavior of the system and can be found from other physical reactions, not only gravitation. In the general case we cannot obtain the general solution of the present equation, but can believe that this equation governs the thermodynamics of self-gravitating systems.

Now we consider one of many simple cases. We assume that there exists a polynomial relation between density and temperature that is given by $\sqrt{\beta(\mathbf{r})}\rho(\mathbf{r}) = A = \text{const}$ and the density changes in accordance with the non divergence of the number of particles that is possible if the density decreases as the fourth power of the distance from center, $\rho(r) = C r^{-4}$. As follows from the normalization condition, $C = N R_c$ where R_c determines the classical limit similarly to the previous case. Such behavior of the density is associated with the real behavior of the gravitational field variable. The equation for the field variable yields the field potential $\varphi(\mathbf{r}) = \frac{r_m A}{r}$ that directly reconstructs the behavior of the gravitational field. As follows from this condition, the temperature also decreases as the fourth power of the distance and hence $\sqrt{\beta(\mathbf{r})} = \frac{A r^4}{C}$ or $T = \frac{C}{A} r^{-8}$. From the equation for energy conservation, we obtain a relation between the constants introduced and the energy of the system, i.e., $\frac{C^2}{A R_0^3} - \frac{A C r_m}{R_0^4} = E$. In this simple case we can obtain all the required coefficients and the spatial dependence of the density and temperature for inhomogeneous particle distributions in a self-gravitating system. This solution can describe the behavior of the temperature and concentration in a real self-gravitational object provided an analysis of the experimental situation is carried out.

CONCLUSION

The self-gravitating systems are non-equilibrium a priori. Indeed, the up-to-date non-equilibrium statistical description considers only probable dilute structures in a self-gravitating system but does not describe metastable states and tells nothing about the time scales of the kinetic theory. The new approach in terms of a non-equilibrium statistical operator with allowance for inhomogeneous distributions of particles and temperature is proposed. The method employs the saddle-point procedure to find the dominant contributions to the partition function and provides a possibility to obtain all the thermodynamic parameters of the system. The statistical operator has no singularities for various values of the gravitational field. The approach makes it possible to solve the problem of self-gravitating systems of particles with inhomogeneous distributions of particles and temperature. Probable specific features of the behavior of a self-gravitating system are predicted for various conditions. The equation of state for self-gravitating system has been determined. A new length of the statistical instability and parameters of the spatially inhomogeneous distribution of particles and temperature are found for real gravitational systems. The gravity factor can either promote or retard such transformations depending on the system and conditions concerned. For the first time a description is given of the formation of spatially inhomogeneous particle distributions accompanied by the changes of temperature. The statistical description of the system is tailored to treat the gravitating particles with regard for an arbitrary spatially inhomogeneous particle distributions. In this approach, the probable behavior of a self-gravitating system can be predicted for any external conditions. In this way we can solve the complicated problem of statistical description of self-gravitating systems. Moreover, the method can also be applied for the further development of physics of self-gravitational and similar systems that are not far from the equilibrium.

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